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Review

Decision-theoretic models of visual perception and action

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ABSTRACT

Statistical decision theory (SDT) and Bayesian decision theory (BDT) are closely related mathematical frameworks used to model ideal performance in a wide range of visual and motor tasks. Their elements (gain function, likelihood, prior) are readily interpretable in terms of information available to the observer. We briefly describe SDT and BDT and then review recent work employing them as models of biological perception or action. We emphasize work that employs gain functions and priors as independent or dependent variables.

At one extreme, Bayesian decision theory allows the experimenter to compute ideal performance in specific tasks and compare human performance to ideal (Geisler, 1989). No claim is made that visual processing is in any sense “Bayesian”. At the other extreme, researchers have proposed Bayesian decision theory as a process model of “perception as Bayesian inference” (Knill & Richards, 1996). We end by discussing how possible ideal models are related to imperfect, actual observers and how the “Bayesian hypothesis” can be tested experimentally.

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1. Introduction

Statistical decision theory (SDT) emerged with the publication of Blackwell & Girshick's *Game Theory and Decisions* in 1954. An immediate stimulus to its development was the *Game Theory* by von Neumann and Morgenstern (1944/1953) and, like game theory, SDT is normative: it is a mathematical method for selecting optimal actions under conditions of uncertainty. On each of a series of turns in SDT a player gains instantaneous information about an uncertain environment and then selects an action. The choice of action determines whether the player merits reward or incurs punishment.

Bayesian decision theory (BDT) is a special case of SDT. Both methods are widely employed in mathematical statistics (Berger, 1985; Ferguson, 1967; Gelman, Carlin, Stern, & Rubin, 2003; Jaynes, 2003; O'Hagan, 1994) and pattern classification (Duda, Hart, & Stork, 2000). In recent years, BDT has been more and more frequently used in developing models of biological perception and action (Knill & Richards, 1996; Maloney, 2002; Mamassian, Landy, & Maloney, 2002; Yuille & Bülthoff, 1996), in part because its mathematical structure resembles the ordinary “perceptual cycle” (Neisser, 1976).

SDT comprises a ‘mathematical toolbox’ of techniques, and anyone using it to model decision making in biological vision must, of

course, decide how to assemble the elements into a biologically-pertinent model. In the following we will first describe the elements of SDT/BDT, then review selected recent work emphasizing these methods, and last discuss the implications of using SDT/BDT as a model of biological perception and action. Earlier reviews include Knill and Richards (1996), Maloney (2002), Mamassian, Landy & Maloney (2002), and Körding (2007).

2. The elements of SDT

The elements of SDT consist of just three sets and three functions. The three sets are \mathbf{W} , the states of the world, \mathbf{X} , the possible sensory states, and \mathbf{A} , possible actions (Fig. 1A). On every “turn”, the world is in some specific state, $w \in \mathbf{W}$, unknown to the observer. The observer is given access to a sensory state $x \in \mathbf{X}$,¹ and must decide what action, $a \in \mathbf{A}$ to select. The interpretation of these elements is very flexible. The state of the world may be the distance to a specific object or the intrinsic color of a surface. Actions could include estimates of depth, a motor program specified as a pattern of neural activity over time, or a decision between fight and flight. Signal detection theory (Green & Swets, 1966/1974) is an application

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¹ We use upper-case x to denote the particular sensory state available to the observer on a specific occasion and lower-case x to denote sensory states in general, the latter analogous to “the people you know”, the former to “your good friend Dennis” who just walked into your office.

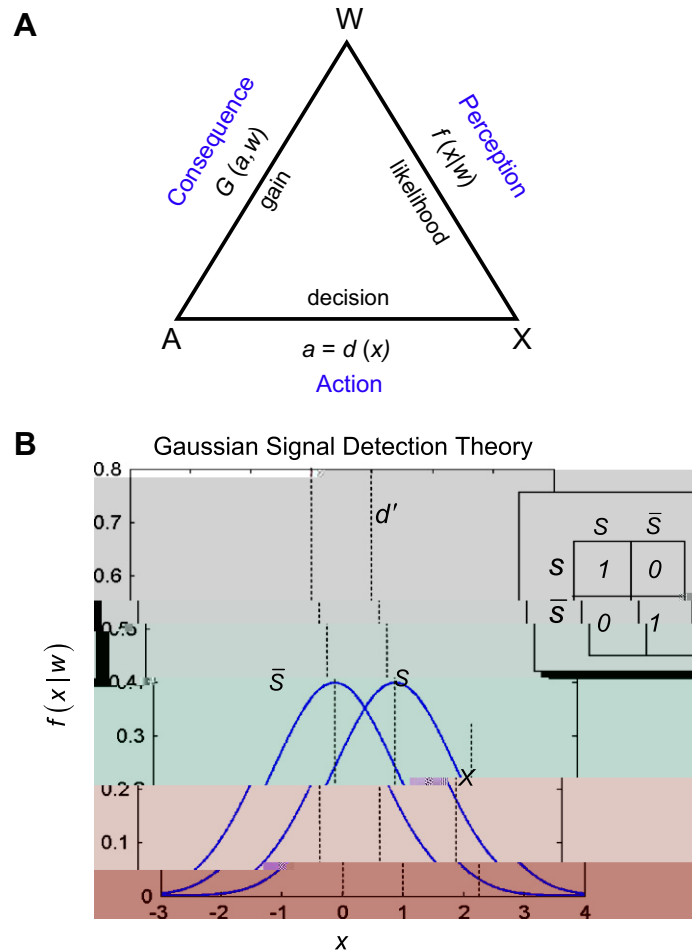


Fig. 1. (A) The three edges correspond to the $G(a, w)$ gain, $f(x|w)$ likelihood, and $a = d(x)$ action. The three vertices correspond to W , the possible world states, X , the possible sensory states, and A , the available actions. (B) The distribution of the sensory state x depends on the state of the world. The two possible world states are S ("signal present") and \bar{S} ("signal absent") and the distributions are Gaussian with equal variance but differing in mean by μ (Green & Swets, 1966/1974).

of SDT that nicely captures all the key ideas and we will use a particular signal detection example to illustrate the key ideas of SDT (and, later, BDT) as we introduce them.

The states of the world in signal detection theory are just "signal present" and "signal absent" denoted as $W = \{S, \bar{S}\}$, the sensory states are any real magnitude that we refer to as the x of the signal $x \in X = \mathbb{R}$ and the possible actions are simply to say "signal present" or "signal absent", denoted as $A = \{a, \bar{a}\}$.

There are three functions that serve to complete the description of SDT (Fig. 1A). The first is the likelihood function $f(x|w)$, the probability density of sensory states contingent on the state of the world which, as written, links the sensory information to the state of the world.² Remarkably, it can be shown that the likelihood function captures all of the sensory information relevant to estimating the state of the world (Berger & Wolpert, 1988; Maloney, 2002), a result known as the Likelihood Principle.

In Fig. 1B we plot the two possible likelihood functions of Gaussian equal variance signal detection theory, one for the world state S ("signal present") and one for the world state \bar{S} ("signal absent"). These are the probability density functions that may have, depending on the state of the world

$$f(x|S) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$f(x|\bar{S}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu-\mu')^2}{2\sigma^2}} \tag{1}$$

where μ' is the mean of the distribution when the signal is present. One possible value of μ' is marked on Fig. 1B and, while it could have arisen from either distribution, it seems intuitively plausible that it arose from the world state "signal present".

The second function is the gain function $G(a, w)$ that determines the gain or loss experienced by the observer on a particular trial. It is also referred to as $U(a, w)$ or $U(a, w)$ in the literature. Losses are just negative gains and vice versa. A possible gain function for the simple signal detection theory model we consider is tabulated in an inset to Fig. 1B. With this gain function, the signal detection theory observer gains one unit if she correctly names the state of the world and otherwise receives nothing.

The third function is the decision function $a = d(x)$ that captures the strategy of any particular SDT observer. The decision function maps the sensory state (the only novel information available on a particular trial) to an action. This modest function is intended to model all of perceptual and cognitive processing. In signal detection theory, the choice of a rule $d(x)$ applied to the signal strength completely specifies the signal detection observer.

We will add one more function, the prior distribution of states of the world, below and, once we do so, SDT will transmute into

² The likelihood function is often written as $L(w|x) = f(x|w)$ to emphasize that it provides information about possible states of the world given a known sensory state x . We will, however, continue to use $f(x|w)$.

BDT, a special case of SDT where the observer has access to the prior distribution of states of the world. For now though, we will consider what we can say about different choices of decision functions in SDT without a prior. One reason to do so is to develop a better understanding of SDT. A second reason is to examine what we can say about different decision rules even when the prior distribution is not known.

We characterize any decision rule (\cdot) by evaluating its expected gain in each world state

$$EG[\cdot|w] = \int_{-\infty}^{\infty} G(\cdot, w) (\cdot|w) \quad (2)$$

The equation is readily interpreted: the state of the world determines the probability density that each possible sensory state can occur through the likelihood function $(\cdot|w)$; the decision function maps the sensory state to an action $= (\cdot)$, and the observer receives the gain $G(\cdot, w)$, weighted by the likelihood summed across all possible sensory events. In the signal detection theory example, with the gain function shown in Fig. 1B, Eq. (2) can be written as

$$\begin{aligned} EG[+|+] &= \int_{-\infty}^{\infty} G(+, +) (+|+) = [+ (+) = +] \\ EG[+|-] &= \int_{-\infty}^{\infty} G(+, -) (+|-) = [+ (-) = -] \end{aligned} \quad (3)$$

and, in the ordinary terminology of signal detection theory, the two rightmost probabilities are the probability of a “hit” (correctly identifying the signal when present) and the probability of a “correct rejection” (correctly identifying the signal when absent) denoted $[H]$ and $[C]$, respectively. We can summarize any decision rule by $EG[\cdot|w]$ and, for the signal detection theory example, we can plot this summary as a plot of $EG[+|+]$ versus $EG[+|-]$. That is, we plot $[H]$ versus $[C]$. The range of expected gain on both axes in this case is 0–1 and gain is synonymous with probability correct. We refer to the resulting plot as a $[H]$ vs. $[C]$ plot and the point plotted for each decision rule as the $[H]$ vs. $[C]$ point for that rule. For any rule (\cdot) we can compute its gains plot but we cannot guarantee that every point on the gains plot has a rule. We plot some examples of gains plots for decision rules as shown in Fig. 2A. The exact location of the gains plot for each rule depends on the likelihood functions in Fig. 1B through Eq. (3).

The rule $r_1(\cdot)$ always chooses the action $+$, the rule $r_2(\cdot)$ always chooses the action $-$, the rule $r_3(\cdot)$ chooses the action $+$ precisely when > 0.5 , and the rule $r_4(\cdot)$ chooses the action $-$ precisely when ≤ 0.5 . The third rule is intuitively appealing. If the sensory state is greater than the point where the two distributions cross in Fig. 1B, we choose $+$ and otherwise $-$. If, however, the world state is certain to be $+$, then r_1 will always earn the maximum possible gain. The rule r_4 in contrast, seems perverse, inferior to the others. As we shall see below, it is.

Given any two rules, say r_2 and r_3 , we can mix them probabilistically by deciding to use r_2 with probability α and otherwise r_3 . We denote the resulting mixture rule as $r_5(\cdot)$. The expected gain for $r_5(\cdot)$ mixture rule in world state $+$ is easy to compute. With probability α we execute rule $r_2(\cdot)$ with expected gain $EG[+|2]$ and otherwise (with probability $1 - \alpha$) we execute rule $r_3(\cdot)$ with expected gain $EG[+|3]$. The overall expected gain for the world state $+$ is just

$$EG[+|5] = \alpha EG[+|2] + (1 - \alpha)EG[+|3] \quad (4)$$

We can similarly compute

$$EG[+|-] = \alpha EG[+|-|2] + (1 - \alpha)EG[+|-|3] \quad (5)$$

The gains plot of the mixture rule r_5 corresponds to a point in Fig. 2A plotted that is on the line joining the points for r_2 and r_3 . Its displacement from r_3 along the line is proportional to α . The point for r_5 is plotted on Fig. 2 under the assumption $\alpha = 0.25$.

The shaded region in Fig. 2A contains the plots of $[H]$ vs. $[C]$, for all possible rules (\cdot) including mixture rules. The top-

right edge of this region, marked by a heavy blue curve, is the receiver operating characteristic curve (ROC curve) of signal detection theory (Green & Swets, 1966/1974) slightly disguised as we have plotted $[C]$ on the horizontal axis rather than the more familiar probability of a “false alarm”. A false alarm occurs when the decision rule selects “signal present” when the world state is “signal absent”, and $[FA] = 1 - [C]$. If we switched to $[FA]$ we would “flip” the plot left to right, restoring the form of the ROC curve that is likely familiar to the reader. In the form we employ, gain increases as we go to the right or up. The unmarked bottom-left side of the region is a sort of anti-ROC curve. If you take any rule on the ROC curve and simply respond $-$ when the rule dictates $+$ and vices versa, you get a rule whose gains plot is on the anti-ROC curve. The rule r_4 is the “anti-rule” to r_3 and vice versa. An observer can only do very badly in a signal detection task if he has the capability to do very well.

2.1. D

The decision rule r_3 always has a higher expected gain $EG[+|w]$ than decision rule r_4 for all states of the world. Consequently, employing r_3 rather than r_4 always leads to a higher expected gain. We say that one decision rule r_3 dominates another r_4 precisely when $EG[+|w] \geq EG[+|w]$ (6)

for all $w \in \mathbf{W}$ and, for at least one choice of w , the inequality is strict. In Fig. 2B we illustrate dominance graphically. All the rules whose plotted expected gains fall into the rectangular area are dominated by the rule whose gains plot falls at the top-right vertex of the rectangle. A decision rule r_4 that is dominated by another rule is r_4 . A decision rule that is not dominated by any other rule is r_3 . The admissible rules in Fig. 2A are precisely those that fall on the top-right frontier marked by a heavy blue curve, the ROC curve. The rules r_1, r_2, r_3 are admissible, r_4 is not and r_5 is not. In the signal detection example, any mixture of two rules with $0 < \alpha < 1$ such as r_5 is inadmissible.

2.2. M

Dominance imposes a partial ordering³ on the decision rules. If one decision rule dominates another then the former offers higher expected gain without further consideration of the state of the world. But any admissible rule neither dominates, nor is dominated by, any other admissible rule. We have no obvious way to choose among the rules whose gain plots fall on the heavy blue curve in Fig. 2A. The

minimax rule allows us to select a rule that gives the “best worst case”. We score each rule by identifying the worst it can do, its minimum gain. For example, the minimum gain for decision rule r_1 (always say “signal present”) is, of course, 0 when the signal is absent, world state $-$. A minimax rule (there may be more than one) has the “best worst case”, that is the “maximum minimum gain”.⁴ The rule r_3 is a minimax rule, an outcome that is not completely surprising when we consider that: (i) it is admissible and (ii) the gains and loss for correct and incorrect responses are identical in the two world states.

The minimax criterion makes no use about any information that we might have about the relative probability of states of the world and Savage (1954) criticizes it and its “worst case” emphasis as

³ A set of items is a complete order if the ordering is complete (every item is either greater than, less than, or equal to, every other item) and the ordering is transitive (if $x > y$ and $y > z$ then $x > z$). A partial order is transitive but need not be complete. That is, some pairs of items may not be ordered.

⁴ SDT and BDT are typically presented with gain functions replaced by loss functions, a cosmetic change if we think of gains as just negative losses and vices versa. Then a minimax rule minimizes the maximum loss and the origin of the term minimax is evident. We retain the term “minimax” even though we work with gain functions.

simple case, any admissible rule is a Bayes rule for some choice of prior and any Bayes rule is admissible. See Maloney (2002) for discussions on more complex BDT models and the mathematical conclusions we can draw from them.

2.4.

The literature concerning Bayesian approaches to biological vision is almost entirely concerned with Bayes rules, rules that have the maximum possible Bayes gain. If we think of an organism as embodying a decision rule then it is appealing to think of the Bayes rule for a given prior as specifying the maximum expected gain possible for any organism when that prior is correct. However, the Bayes criterion can also be used to order rules (organisms) that are distinctly sub-optimal. All of the decision rules sharing a single dashed red line in Fig. 2C have the same Bayes gain and Bayes gain increases as the red dashed line moves to the right. To compare two rules (organisms) we need only determine the line each is on and then determine which line is more to the top-right. We'll return to this point in a later section, I

2.5.

We simplified the signal detection example by choosing a very simple and symmetric gain function (Fig. 1B, inset). A different choice of gain function would only transform the axes of Fig. 2C by a linear transformation.⁶ In Fig. 2D, we replot the gains plot if we set $G(+, +) = 0.5$ but keep $G(+, -) = 1$ and $G(-, -) = G(-, +) = 0$. The result is a compression by 0.5 along the vertical axis. The prior vector and the equivalent Bayes lines are unaffected and consequently the Bayes rules in 2C are no longer Bayes rules. We have shifted to a rule that puts more emphasis on correctly classifying the absence of a signal. This outcome is intuitive since correctly classifying the absence of a signal is worth twice as much as correctly classifying the presence of a signal.

3. Modeling biological perception and action

As just presented, SDT and BDT are mathematical frameworks that can be used to model biological performance in perceptual-motor tasks and such models have been widely employed in the study of perception and action over many decades (see for discussion, Geisler, 1989; Landy, Maloney, Johnston, & Young, 1995). The visual cue combination literature, for example, compares human performance in visual estimation tasks to model observers that minimize squared error (variance). No short review could encompass this very large and important literature and much of this early work does not systematically vary the elements of SDT/BDT.

In this section we instead review recent experimental studies that test models of perception and action based on SDT/BDT that systematically vary elements of SDT/BDT: prior, likelihood, gain. The first set of studies, for example, test whether human observers can plan movements to maximize expected gain with arbitrary gain functions imposed on the outcomes of possible movements. Each of these studies can be viewed in two ways: as a comparison of human performance to that of an idealized counterpart that makes perfect use of the perceptual and motor capabilities of the organism, or as a process model of the process by which the visuo-motor system carries out the task. We return to this point below.

⁶ If the slope of either linear transformation is negative then we have chosen a gain function that encourages observers to make errors. E.g. we pay the observer more for false alarms than for correct rejections. The discussion in the text assumes that we have chosen gains that lead to positive slope parameters.

3.1. A

Trommershäuser and colleagues report a series of experimental tests of whether human observers can cope with arbitrary gain functions in a simple visuo-motor task (Trommershäuser, Maloney, & Landy, 2003a,b, 2008). On each trial, the stimulus configuration, composed of one or two red circles and a green circle (Fig. 3A), was presented at a random location on a computer touch screen. The orientation of the target configuration varied from trial to trial as well. After its appearance, the observer had to reach out and touch the screen within 700 ms. The observer received monetary rewards or penalties based on the outcome of his reach. If the observer was late in hitting the screen, he incurred a large penalty. If he touched inside the green circle within the time limit he earned a reward (100 points), but, if within the red circle, he incurred a penalty that varied with experimental condition. A hit within the region where the two circles overlapped incurred both the reward and the penalty and if the participant hit the screen outside of both circles within the time limit, he received nothing.

The observer could decide where to aim but could not completely control where he hit. A speeded movement aimed at the center of the green circle had a substantial probability of missing the green circle altogether because of the observer's intrinsic visual and motor uncertainty. Before the formal experiment, the observer had practiced the movement for several hundred trials. The observer was rewarded for hitting within the green circle but no penalty was imposed for hitting within the red. Training continued while the observer learned to respond within the time limit, minimized his own motor error, and maximized his probability of hitting within the green circle.

In the main part of the experiment, the experimenter imposed penalties for hitting within the red circle as described above. The observer faced a decision problem that had the same prior and likelihood functions as during training but with different gain functions specified by the penalties and the spatial arrangement of circles.

In the experiment of Trommershäuser et al. (2003a), the relative position of red and green circles varied from trial to trial with six different horizontal displacements. The six gain functions were interleaved. For each observer and condition, the aim point that maximized expected gain (e.g. the white spot in Fig. 3A) was distinct. The observer's mean end point in each condition could therefore be compared with the aim point that maximized his expected gain. The comparison for all observers is shown in Fig. 3B. The observers' performance shows no obvious deviations or trends from that that would maximize expected gain as predicted by BDT.

One possibility is that observers in the decision task gradually improved their aim in response to penalties and rewards. If so, we would conclude only that observers could maximize their expected gain by a gradual "hill-climbing" process driven by reinforcements.

To test this possibility, Trommershäuser et al. (2003a) examined the displacements of end points away from the center of the green circle along the axis joining the centers of the red and green circles (the white line in Fig. 3A). These are shown in Fig. 3C for one observer with 0 on the vertical scale corresponding to the mean displacement across all trials with the stimulus configuration shown to the right. If observers only gradually learned the aim point that maximized expected gain, we would expect to see trends in the early part of these plots. There were no evident patterned trends across the first few trials (Fig. 3C) and the correlation between successive trials was not significantly different from 0 (possibly because all stimulus configurations were randomly interleaved).

The implications of Trommershäuser et al. (2003a,b, 2008) are, first of all, that people either learn their own visuo-motor spatial

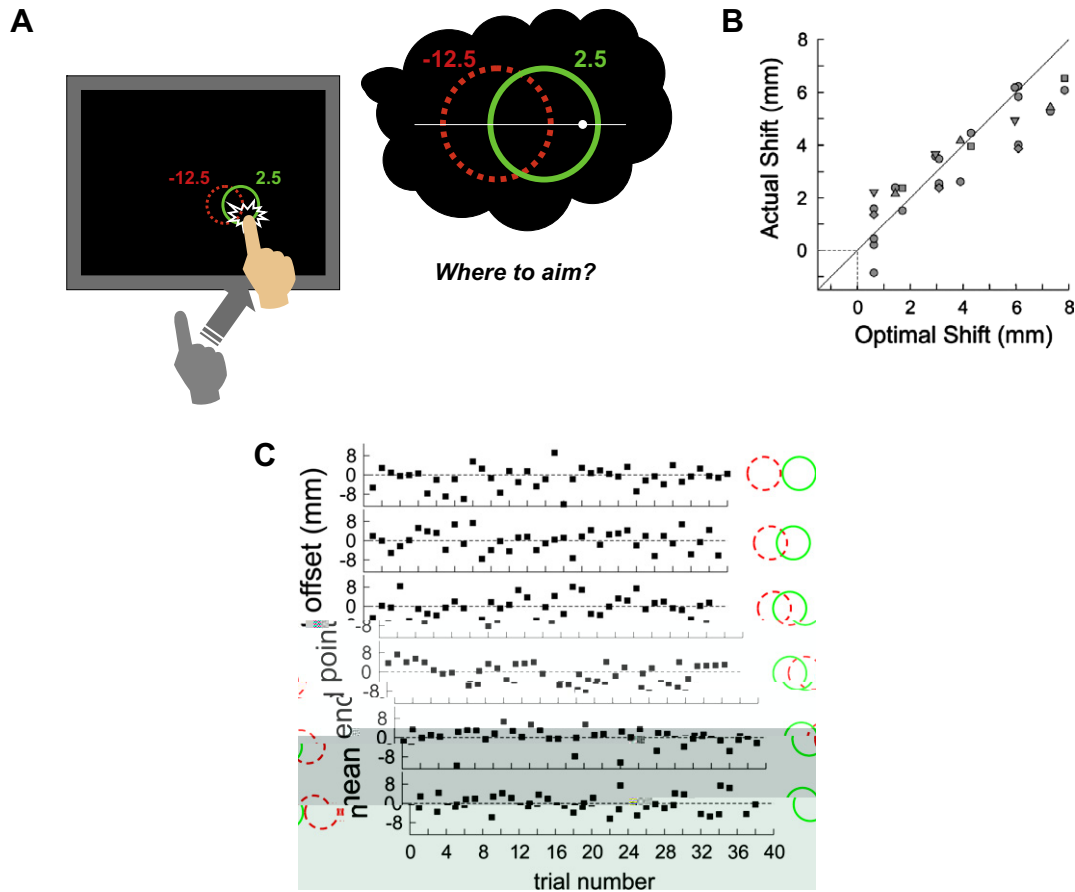


Fig. 3. (A) A stimulus configuration such as the one shown appears on a computer screen in front of the observer who was instructed to reach out and touch the screen within 700 ms. The gain function is coded by colored circles whose position and relative orientation change from trial to trial. A hit within the solid green circle results in a gain of 2.5 cents, within the dashed red circle, a loss of 12.5 cents. The observer moves rapidly and cannot completely control his movement. Even if he aims at a particular point on the screen the result is a probability distribution of actual endpoints which induce probabilities of hitting within each region. A possible aim point is marked by a white dot. How much should the observer aim away from the dashed red circle to maximize expected gain? (B) Actual choice of aim point (horizontal deviation along the white line) plotted versus optimal choice of end point computed via BDT. (C) Trial-by-trial deviation of movement end point (in the horizontal direction) a function of trial number after introduction of rewards and penalties for six different gain functions. Figure reproduced with permission from Trommershäuser et al. (2008).

uncertainty spontaneously during training or, less likely, that they knew it before the experiment began. Second, they could combine their knowledge of visuo-motor uncertainty with novel gain functions to choose the aim point that maximizes their expected gain.

The gain function for any task determines the possibilities for reward and punishment; it is remarkable that observers in the tasks of Trommershäuser et al. could come close to maximizing expected gain with such arbitrary choices of gain function.

People do fail in similar tasks when the gain functions are more complex. Wu, Trommershäuser, Maloney, and Landy (2006) pointed out that the stimuli and gain functions used in Trommershäuser et al. (2003a,b) were always symmetric around a line joining the centers of the red and green circles (Fig. 3A, inset). Observers may have deduced that the optimal aim point always lay on this line and this insight may have aided them in planning movement.

Wu et al. (2006) used stimuli with a reward region and two penalties regions differing in magnitude of penalty and found that observers showed patterned failures in selecting aim points. They tended to regress toward the line of symmetry.

3.2. C

Körding and Wolpert (2004) asked observers to reach out and touch a target. Their movement drove a cursor onto a visual target

and nominally the cursor corresponded to the location of their index fingertip. Observers were never allowed to see the hand they reached with. On some trials, the cursor was laterally displaced relative to the actual position of the fingertip. On each movement, the lateral shift of the cursor was randomly drawn from a Gaussian distribution with a mean of 1 cm to the right of the finger and a standard deviation of 0.5 cm. There were four feedback conditions (Fig. 4A). In the σ_0 condition, the position of the cursor was signaled by a white dot whose uncertainty simply reflected the observer's own visuo-motor error. In the σ_M or σ_L conditions, extra uncertainty was introduced by using a cloud of dots with medium (σ_M) or large (σ_L) standard deviation to mark the nominal location of the fingertip. In the final, σ_∞ condition, feedback was withheld. In all conditions, feedback was presented briefly when the fingertip was halfway to the target (Fig. 3A). The endpoint of the reaching movement was presented only for the σ_0 condition.

The question that Körding and Wolpert (2004) addressed was how much the observer should compensate for uncertain visual feedback. Suppose that, on a specific trial, the observer sensed a lateral shift 2 cm to the right. The true lateral shift might be 1.8 cm or 2.2 cm to the right, but the former possibility was more likely than the latter given that the shift was drawn from a prior that was Gaussian with mean 1 cm. Intuitively, the observer's compensation for the 2 cm error should regress toward 1 cm and the degree of regression depends on condition.

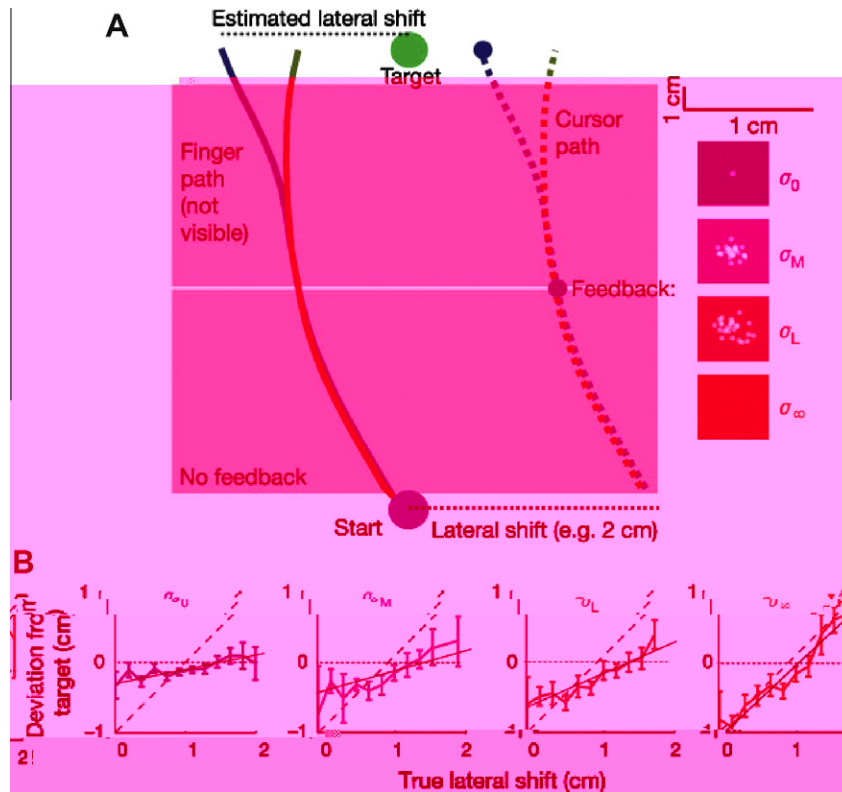


Fig. 4. (A) Observers reached out to move a cursor onto a visual target. They never saw their hand. The cursor was horizontally displaced away from the actual position of the finger by a random distance that had a Gaussian distribution with a mean of 1 cm to the right and a standard deviation of 0.5 cm. Halfway to the target, a visual feedback of the cursor was briefly provided with no extra uncertainty (σ_0), medium extra uncertainty (σ_M), large extra uncertainty (σ_L), or withheld (σ_∞). (B) The mean lateral deviation of the cursor at the end of the movement plotted against the true lateral shift for a typical observer. Solid lines denote the fit of a Bayesian observer model, whose slope indicates the relative weights of prior and likelihood functions. The higher the uncertainty, the more weight the observer put on the prior. Figure reproduced with permission from Körding and Wolpert (2004).

This trend was observed in human observers. In Fig. 4B, the mean deviation of the endpoint of the cursor from the target is plotted against the true lateral shift for a typical observer for each of the four conditions. If the sensed lateral shift were fully compensated, the mean deviation should have been 0 and the results for the σ_0 are not far from this limiting case. In contrast, in the σ_∞ condition, the observer failed to compensate or nearly so. As Fig. 4B shows, the higher the uncertainty, the more weight on the prior, the larger the slope. Note that, except for the σ_0 condition, the observer had no opportunity to progressively learn the appropriate weight for a specific feedback condition, because no feedback was provided for the final position of the finger.

Possibly as a consequence, the likelihood functions inferred from the Bayesian observer model did not agree with the actual likelihood functions. The observers' estimates of the standard deviations of the halfway visual feedback were inferred to be 0.67 and 0.8 cm, respectively for the σ_M or σ_L conditions, much smaller than the actual standard deviations, 1 and 2 cm. The picture that emerges is that of an observer whose performance changes from condition to condition in qualitative agreement with BDT but who is effectively using erroneous estimates of likelihood functions.

3.3.

Several studies have focused on whether people could choose the temporal parameter that maximizes expected gain of the movement. In Battaglia and Schrater (2007), observers reached out to touch an invisible target that was indicated by a cloud of dots whose positions were randomly drawn from a two-

dimensional Gaussian distribution (Fig. 5A). The observer would receive a monetary reward if he successfully touched the target within a specific time limit. From the start of a trial, the dots appeared one by one across time until the observer initiated the movement. Therefore, the longer the observer waited to move, the more dots he would see, and the more accurate the visual estimate of target location. But increased viewing time came at the expense of a reduction in time available to carry out the movement and a consequent increase in spatial variability dictated by the observer's speed-accuracy tradeoff in movement. With the visual and motor uncertainties measured in separate control tasks, Battaglia and Schrater modeled the probability of touching the target as a function of viewing time (t_v) and movement time (t_m). There were three scatter levels of dots, low, medium, and high, leading to low, medium, and high levels of uncertainty of target position given the same number of dots. The temporal parameters observers chose were close to those of the model that maximized their expected gain and varied with the experimental conditions in the correct direction. As illustrated by Fig. 5B, the higher the uncertainty associated with dot scatter level, the more time allocated to viewing.

In the task of Battaglia and Schrater (2007), the time allocation influenced the consequence of the movement in an indirect way. Even so, human observers chose the movement parameter that almost maximizes their expected gain. It implies that people know how their spatial accuracy changes as a function of movement time and are able to combine this knowledge with novel gain functions. Similarly, Dean, Wu, and Maloney (2007) showed that when the reward of reaching a target decreases linearly with movement time, people choose the movement time that nearly maximized their expected gain. However, a surprising failure was found when

observers attempted to touch two targets one after another within an overall time limit (Wu, Dal Martello, & Maloney, 2009; Zhang, Wu, & Maloney, 2010). For example, in Fig. 6, the observer first touched the blue circle then the green circle. The two targets corresponded to the same (left) or different (right) rewards. Allocating more movement time to a target would plausibly increase the probability of hitting it and earning the corresponding reward. However, even when the second target was five times more valuable than the first target, observers still allocated slightly more time to the first target.

3.4. A

A recent study considered tasks analogous to those of Trommershäuser et al. (2003a,b, 2008) but with gain functions that are specified in the temporal domain (Hudson, Maloney, & Landy, 2008). Before the formal experiment, observers completed extensive training of reaching at specific movement times. During this initial training period, observers attempted to make movements of specified durations to hit targets on a computer touch screen (Fig. 7A). Prescribed times were specified on a time bar and, after every trial, the observer's actual duration was also plotted on the time bar so that observers could compare their time to the prescribed time and improve their training performance.

In the main experiment, observers saw a specification of a temporal gain function. Fig. 7B shows the four temporal gain functions used in the experiment. Their task was to plan a movement to hit the target at a time of their choosing. The planned duration of their movement controlled their expected gain and their performance was compared to performance maximizing expected gain.

The actual versus optimal movement times across the four conditions and all observers are summarized in Fig. 7C. Observers were close to optimal. No obvious trends of learning were identified. Hudson et al. compared observed performance across time to reinforcement learning models and excluded the possibility that such models predict observed performance: "To investigate the possibility that observers used a hill-climbing strategy during the main experiment, instead of maximizing expected gain by taking account of their own temporal uncertainty function and experimentally imposed gain function, we performed a hill-climbing simulation using each observer's temporal uncertainty function. In the simulation, intended duration was moved away from the penalty region by 3Δ ms after each penalty and towards the center of the target region by Δ ms for each miss of the target that occurred on the opposite side from the penalty (corresponding to the 3:1 ratio of penalty to reward). The value of Δ was initially set to be relatively large. With each change of direction of step, Δ was reduced by 25% to a minimum step size of 1.5 ms. While this simulation approximately reproduced the final average reach times observed experimentally, it does not provide a good model of observer performance. First, there were significant autocorrelations of reach durations beyond lag zero in the simulation data but not in the experimental data. Second, a learning algorithm would be expected to produce substantially higher σ values during test than those observed during training. This is what we found with our hill-climbing simulation. Using observers' training σ values to produce the simulated data, the simulation produced 17 out of 20 main-experiment σ values that were above the training values, whereas our observers' main-experiment σ values . . . were entirely consistent with temporal uncertainty functions measured during training." (Hudson et al., 2008, pp. 4–5).

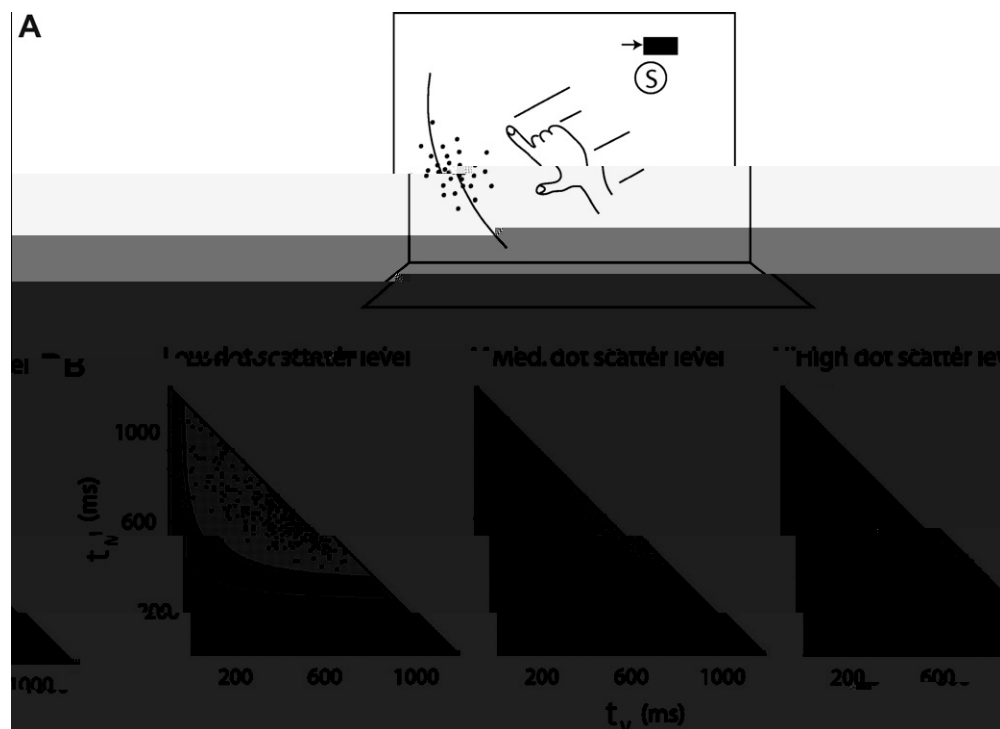


Fig. 5. (A) An invisible target was indicated by the surrounding cloud of dots whose positions were drawn from a two-dimensional Gaussian distribution. The observer started from the "S" to reach the invisible target within a time limit. Successful hit resulted in monetary reward. The black bar indicated the amount of time that had elapsed from the onset of the trial. The dots indicating the target accumulated with time at a constant rate until the movement started. The observer had to make a tradeoff between viewing time and movement time. (B) Movement time plotted against viewing time for a typical observer. Each dot denotes a trial. Contours denote the expected gain of the observer predicted by the model. The three scatter levels of dots, low, medium, and high, led to low, medium, and high levels of uncertainty of target position given the same number of dots. The temporal parameters observers chose were close to optimal and varied with the dot scatter level in the correct direction, that is, the higher the uncertainty associated with dot scatter level, the more time allocated to viewing. Figure reproduced from Battaglia and Schrater (2007).

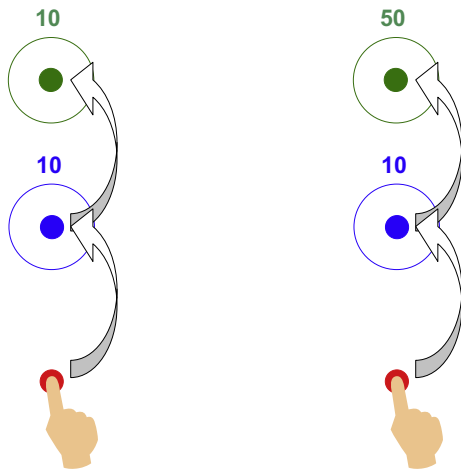


Fig. 6. A . The observer first touched the blue circle then the green circle. The two targets corresponded to the same (left) or different (right) rewards. The movement time allocating to a target would increase its probability to be hit and thus its reward be won. However, even when the second target was five times valuable than the first target, observers still stuck to allocate slightly more time to the first target. Figure reproduced from Zhang et al. (2010). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

In contrast, Mamassian (2008) found that observers in a timing task also failed to maximize expected gain but in this task observers had no prior training with the task and therefore less information about their own timing uncertainty. Maloney and Mamassian (2009) discuss the possible effect of training on observers' abilities to maximize gain.

3.5. \int \int

Balci, Freestone, and Gallistel (2009) used a clever design to probe how well mice could cope with their own temporal uncertainty. Fig. 8A gives an illustration of the task. On any trial a food reward would appear at one of two separate feeding hoppers. There were two type of trials. On short-trials, the reward was delivered at the "S" hopper with a short latency (3 s) after the start of the trial. On long-trials, at the "L" hopper with a long latency (6 s).

If the mouse stayed at the right hopper until the time of reward delivery, it would get the reward and otherwise nothing. The difficulty for the mouse was to decide when to move from the "S" hopper to the "L" hopper and its performance was limited by its own timing uncertainty. If, for example, a short-latency reward was delivered but the mouse had switched from the "S" hopper to the "L" hopper at 2 s, it would lose the reward due to a w . The opposite error was called a w . Due to its uncertainty in estimating elapsed time, the mouse could not completely determine the actual time of switch. If the mouse chose to switch after it judged 3 s had elapsed then would likely incur a considerable risk of an early switch. On many trials the mouse would judge that 3 s had elapsed when in fact less than 3 s had elapsed and short-latency reward was still possible. The mouse would forfeit the possibility of reward on all such trials. The choice of switch time that maximized expected gain was determined by the mouse's temporal uncertainty.

A counterpart task was carried out on human observers in which observers were rewarded in points for pressing down one of two keys at the time of reward delivery. One key offered a short-latency reward, the other a long, and the human observer could only press one key at a time. The problem for the human observer was to choose the point in time to switch from the

short-latency reward key to the long-latency reward key. The short- and long-latencies for human observers were 2 s and 3 s.

Both human and mouse observers completed several sessions. The probability of short-trials in a session could vary from 0.1 to 0.9. Based on an observer's temporal uncertainty, Balci et al. used a BDT model to solve out the aimed switch time that maximized expected gain for the specific observer. Fig. 8B and C plot optimal switch time against mean switch time for each observer and probability condition respectively for humans and for mice. The average absolute difference between actual and optimal switch times was 172 ms for humans (5% of the 3-s range), and 436 ms for mice (7% of their 6-s range). For mice, Fig. 8D shows how the mean switch time \hat{t}_0 varied almost in the same way with the short trial probability as the optimal switch time \hat{t}_0 did.

For both human and mice observers, this near optimal performance was unlikely to be the result of reinforcement learning. Analyzing trials across the duration of the experiment, Balci et al. identified no discernible improvement (Fig. 8E).

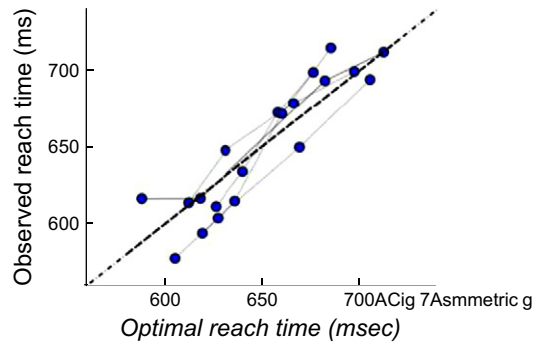
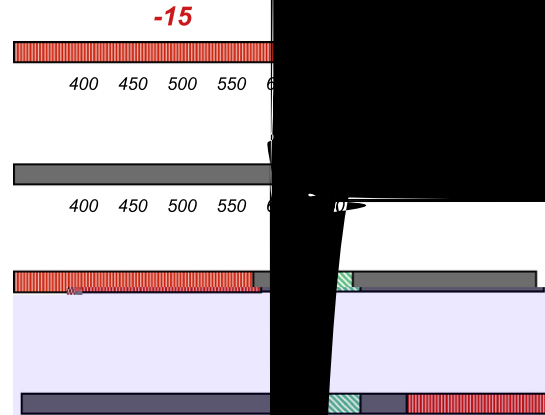
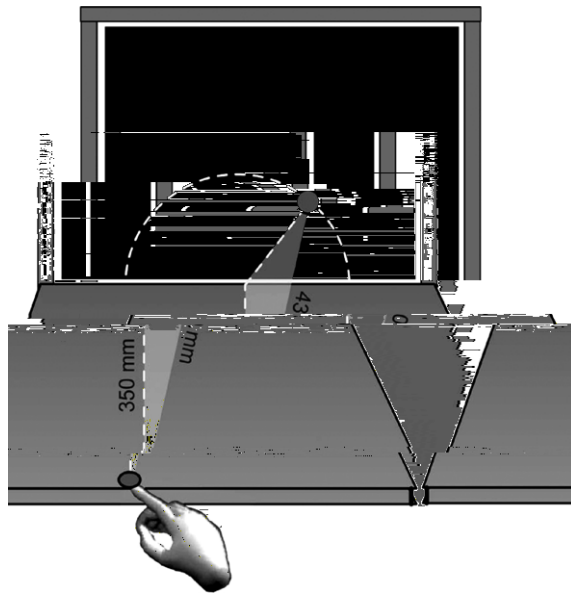
4. Imperfectly optimal observers

Use of BDT as a benchmark model does not imply that human visuo-motor processing is in any sense Bayesian inference, even when human performance is close to ideal (Maloney & Mamassian, 2009). We can view the experimental studies just described as comparisons of human performance to the performance of a BDT observer with the same sensory and motor limitations as the human observer. Geisler (1989) proposed using statistical models as benchmarks in just this way: "... the ideal discriminator measures all the information available to the later stages of the visual system ..." (Geisler, 1989, p. 30). Thus, we compare human performance to a BDT observer precisely because the BDT observer makes the best use of the available information. This benchmark approach grew out of earlier work by Barlow and colleagues (Barlow, 1972, 1995) and it has proven to be a useful tool in the study of human perception (see, for example, Najemnik & Geisler, 2005).

Nevertheless, suppose that we have benchmarked human performance in a visuo-motor task and it is remarkably close to that of its BDT counterpart and we cannot reject the hypothesis that the BDT observer is an accurate model of human visuo-motor processing. This was the outcome of several of the studies we reviewed above. Are we justified in advancing the BDT observer as a model of the perceptual process, at least for this task?

One evident reason that we cannot is technical: a failure to reject the hypothesis of optimal performance may simply be a Type II Error in statistical terms. It is possible that the null hypothesis of optimality is not true but that our experimental design and statistical analyses failed to detect the discrepancy between human performance and ideal. The underlying problem is that the BDT observer is an idealization akin to the notion of a fair coin that has probability of coming down heads of exactly 0.5. No physical coin is ever perfectly fair, and every biological organism can have an off day. In speaking of a fair coin, Feller (1968) justifies the use of such models: "... we preserve the model not merely for its logical simplicity, but essentially for its usefulness and applicability. In many applications it is sufficiently accurate to describe reality." (Feller, 1968, p. 19). However, if the only "reality" we have to describe is that the human observer, in some visuo-motor tasks, does nearly as well as he can be expected to, then there is no reason to conclude that the elements of BDT correspond to anything in human visuo-motor processing. As every child learns in kindergarten these days, there are many ways to be excellent.

A second, and separate problem with BDT as a process model is that the BDT observer needs access to accurate information about likelihood, gain, and prior. In particular, the prior distribution of a



BDT observer is readily interpreted as claims about the environment and the use of the prior is characteristic of Bayesian approaches. Nakayama and Shimojo (1992) argue that the amount of information in the prior for many simple visual tasks is possibly large. Maloney (2002) estimates the number of world states (the size of the domain of the prior) for one simple shape from shading task and shows that it is too large to be plausibly learned from experience or represented neurally.

Nakayama & Shimojo's argument is apparently compelling but there is an evident way out of this difficulty. We need only drop the requirement that the visuo-motor system have exact estimates of priors. The resulting model observer follows the computations of BDT may not maximize expected gain because of its erroneous estimates of prior, gain or likelihood function (Maloney, 2002). We refer to them as ⁷ echoing the title of Janetos and Cole (1981).

In Fig. 9, for example, we consider the case of an imperfectly optimal signal detection observer with an erroneous estimate of the prior. The black arrow is the correct prior vector, the red arrow is the erroneous estimate. The observer selects the Bayes rule $\tilde{\pi}$ dic-

tated by the erroneous prior rather than the true Bayes rule π . With respect to the true prior, the rule $\tilde{\pi}$ is just another sub-optimal rule whose equivalent Bayes gain corresponds to the red dashed line. The maximum possible Bayes gain corresponds to the black dashed line passing through the gains of the Bayes rule. The cost of the observer's error is proportional to the distance between the red and black dashed lines. It is evident that, depending on the discrepancy between the true and assumed priors, the loss in Bayes gain may be small or large and, if small, that the consequences of the error to the observer are slight.

Moreover, consider the minimax rule which assumes nothing about the prior. We can see that the minimax decision rule has higher Bayes gain than the rule $\tilde{\pi}$. In this case, the observer would be better off discarding his "knowledge" about the prior and using the minimax rule instead. However, we can also see that for a wide range of choices of erroneous prior, the resulting erroneous rule actually has higher Bayes gain than the minimax rule. If, for example, the observer were confident that $\pi > 0.5$ but otherwise ignorant of π , he would do well (in terms of Bayes gain) to use a possibly erroneous estimate of prior rather than the minimax rule.

Moreover, recall that BDT allows us to do more than compute the optimal Bayes rule. We can also order any two imperfectly

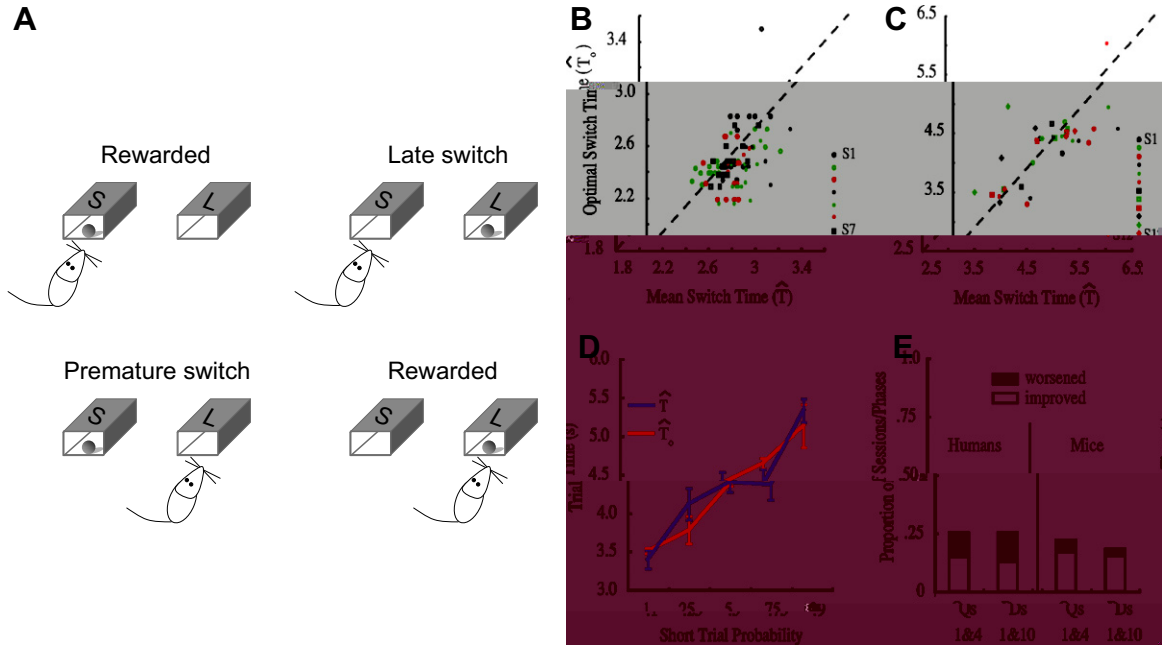


Fig. 8. (A) An illustration of the temporal decision task for mice in Balci et al. (2009). S and L denote two separate feeding hoppers. In any trial, a food reward was delivered either at S with a short-latency (short-trial) or at L with a long-latency (long-trial). If the mouse stayed at the right hopper at the time of reward delivery, it would get the reward, else it would miss the reward. Starting from the S hopper, the mouse might lose its reward for early switch when the reward was at L , or for late switch when the reward was at S . Due to its uncertainty in temporal perception, the mouse could not completely determine the actual time of switch. The mouse needs to decide what time to switch. (B and C) Optimal switch time against mean switch time for each observer and probability condition, respectively for human and mice. (D) Mean switch times (\hat{T}) and optimal switch times (\hat{T}_o) of mice as functions of short trial probability. Error bars denote ± 1 standard error. (E) The proportion of experimental conditions in which performance improved significantly (white bar) or worsened significantly (black bar) between the first and last quartiles or first and last deciles in the sequence of trials within a given condition. Figure (except A) reproduced from Balci et al. (2009).

optimal observers by determining which of the two has the higher Bayes gain with respect to the true prior. We can potentially measure the true prior for any task describable by BDT and also measure the observer's prior experimentally (e.g. Adams, Graf, & Ernst, 2004; Mamassian & Landy, 2001) and work out the cost to the observer of any error in estimating priors. Similarly, Körding

and Wolpert (2004), described above, estimated likelihood functions from human performance and found they were discrepant from actual likelihood functions.

Janetos and Cole (1981) in an article entitled "imperfectly optimal animals", pointed out a third problem with using idealized models as models of biological performance. They described two tasks where animals' performance well approximated the performance of an optimal algorithm similar in spirit to BDT as presented here. They then pointed out that, for both tasks, there was a very simple behavioral rule that would approximate the performance of the optimal algorithm. The experimenter might mistakenly conclude that an organism implementing the simple rule was an instantiation of the optimal rule.

5. Testing the Bayesian hypothesis

The Bayesian approach is not a specific falsifiable hypothesis but rather a (mathematical) language that allows us to describe the structure of the environment, the flow of visual processing, the planning of action. It is a powerful language and therein lies a difficulty. After the data are collected it is not very difficult to develop a Bayesian model that accounts for it. Indeed, almost any applications of Bayesian methods to perception and action are post hoc fitting exercises. If Bayesian models are to be judged useful, then, they must also permit prediction of experimental outcomes, quantitatively as well as qualitatively.

In the discussion of the imperfectly optimal observers, we argued that it was reasonable to expect the prior embodied in a biological observer to be discrepant from the true objective prior and consequently, an observed discrepancy between the prior on estimated from experimental data and the true prior on in the world is not conclusive evidence against the Bayesian approach. However, if we find ourselves estimating the same prior on in

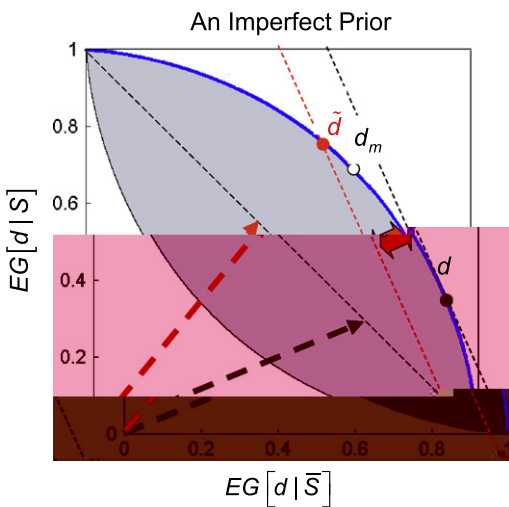


Fig. 9. The consequences of error in choice of prior. The true prior vector is shown in black and the corresponding Bayes rule is \tilde{d} , marked with a black dot. If the observer uses the erroneous prior vector shown in red and the corresponding decision rule, \tilde{d}_m , that would be a Bayes rule for this prior vector, then the cost of using the erroneous prior to the observer is the difference between the two dashed lines (marked by a red block arrow). Use of the minimax rule \tilde{d} would lead to higher expected gain than the decision rule \tilde{d}_m based on the erroneous prior. For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

two different experiments, and find that the two estimates are discrepant, then there are serious grounds to question the entire Bayesian enterprise. We refer to this criterion as a *test of consistency* and it is evidently a test of whether human behavior is controlled by a system of consistent priors on states of the world.

Similarly, Maloney and Mamassian (2009) describe a different test of BDT that they refer to as a *transfer test*. The test assesses whether the visuo-motor system can store and recall and retrieve priors, likelihood functions, and gain functions independently of one another. Maloney and Mamassian argue that the ability to transfer prior information acquired while learning one task to another task carried out later in the same environment would suggest that prior can be stored and reinstated independent of particular tasks.

Consider, for example, the tasks of Trommershäuser et al. (2003a,b) and that of Hudson et al. (2008). In both cases observers had the potential to learn their own spatial and temporal motor uncertainty in training tasks with gain functions different from those employed in the main experiments. This uncertainty was in effect a prior distribution dictating how a movement aimed at a particular point in space and time would be realized.

In the main part of the experiment, they were challenged with a variety of arbitrary gain functions but the movement and its uncertainty were unchanged. That is, the training task and the main experimental task shared the same prior. The lack of any trends in performance in the main experiments of Trommershäuser et al. (2003a,b) and Hudson et al. (2008), indicate that observers could recall and combine prior information learned during training with novel gain functions. Balci et al. (2009) also found no evident trends in performance. Observers in these tasks successfully transferred prior information from one task to a second, passing the transfer test of Maloney and Mamassian (2009).

6. Conclusion

Statistical decision theory (SDT) and Bayesian decision theory (BDT) are mathematical frameworks that are particularly congenial to describing the kinds of tasks biological organisms engage in (Milner & Goodale, 1995). Both theories emphasize the potential gain or loss associated with the outcomes of actions and both emphasize the constraints on action introduced by uncertainty. They provide a natural vocabulary for crafting idealized counterparts to actual observers in order to compare human performance to the best performance possible for the human observer.

In this review we first presented the elements of SDT and BDT and then discussed recent work that systematically manipulates these elements as part of an experimental design. The overall conclusion we can draw is that human observers can exploit arbitrary gain functions imposed on the environment and compensate at least in part for changes in environmental priors. They do not always maximize Bayes gain but, in many experiments, they come remarkably close without obvious pattern in their failures. In other experiments (e.g. Zhang et al., 2010) they fail, sometimes dramatically.

By varying gain functions as an independent variable, we potentially observe a wider range of behavior than we would otherwise observe. Moreover, the pattern of failures and successes observed may aid us in developing accurate process models of human visuo-motor processing.

We also discussed interpretations of BDT as process models of human performance (“perception as Bayesian inference,” Knill & Richards, 1996). We asked in effect whether the elements of BDT (priors, etc.) were useful components of process models of human visuo-motor processing and noted that very simple, non-optimal models can well approximate ideal performance (Janetos & Cole,

1981). We focused on a class of model observers that we referred to as *Bayesian observers* that implement BDT but with possibly erroneous estimates of prior, gain, and likelihood functions. We discussed two methods for testing such whether human performance is captured by an imperfectly optimal observer: comparison tests and transfer tests (Maloney & Mamassian, 2009).

SDT and BDT are fundamentally about combining information about uncertainty and gain so as to maximize the expected gain of the observer. This topic is also central to the study of human decision making. It is interesting to compare human performance in visuo-motor tasks which is often found to be near-optimal to that observed in decision making experiments where decision makers typically do not maximize gain (e.g. Kahneman & Tversky, 2000). The observed deviations are large and patterned, with observers typically showing distortions in their use of both probability and gain. One study (Wu, Delgado, & Maloney, 2009) directly compared human decision making with a mathematically equivalent motor task and found that human observers distort probability in both tasks but that the distortions were markedly different in the motor and “classical” decision tasks.

In summary, the kinds of experiments inspired by SDT/BDT are powerful tools for exploring the limits of human visuo-motor capability and the SDT/BDT framework allows comparison of human performance in apparently different tasks such as decision making and movement planning.

Acknowledgments

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